A METHOD FOR ESTIMATING THE PROPERTIES OF A

SOLID MATERIAL SUBJECTED TO COMPRESSIONAL FORCES

TO ALL WHOM IT MAY CONCERN:

BE IT KNOWN THAT ANDREW J. HULL, citizen of the United States of America, employee of the United States Government and resident Newport, County of Newport, State of Rhode Island, has invented certain new and useful improvements entitles as set forth above of which the following is a specification:

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3 A METHOD FOR ESTIMATING THE PROPERTIES OF A

4 SOLID MATERIAL SUBJECTED TO COMPRESSIONAL FORCES

5

6 STATEMENT OF GOVERNMENT INTEREST

7 The invention described herein may be manufactured and used

8 by or for the Government of the United States of America for

9 governmental purposes without the payment of any royalties

10 thereon or therefore.

11

12 BACKGROUND OF THE INVENTION

13 (1) Field of the Invention

14 The present invention relates to a method to measure (or

15 estimate) the complex frequency-dependent dilatational and shear

16 wavenumbers of a single slab of material subjected to large

17 static compressional forces. More particularly, this invention

18 provides a method to determine complex dilatational wavespeed,

19 complex shear wavespeed, complex Lamé constants, complex Young's

20 modulus, complex shear modulus, and complex Poisson's ratio.

21 (2) Description of the Prior Art

22 Measuring the mechanical properties of slab-shaped materials

23 are important because these parameters significantly contribute

24 to the static and dynamic response of structures built with such

25 materials. One characteristic that most elastomeric solids

1   possess is that, when they are subjected to large static forces

2   (or pressure), their rigidity changes. Materials that have one

3   set of mechanical properties at a pressure of one atmosphere can

4   have very different properties when subjected to increased

5   pressure. The ability to determine the pressure dependence of

6   material properties is extremely important for modeling the

7   behavior of systems comprised of these materials.

8   Resonant techniques have been used to identify and measure

9   longitudinal and shear properties for many years. These methods

10 are based on comparing measured eigenvalues to modeled

11 eigenvalues and calculating the resulting material properties.

12 These methods do not account for static pressure or large

13 compressive forces. Additionally, they typically require long,

14 slender materials to perform the measurement process. Comparison

15 of analytical models to measured frequency response functions are

16 also used to estimate stiffness and loss parameters of a

17 structure. When the analytical model agrees with one or more

18 frequency response functions, the parameters used to calculate

19 the analytical model are considered accurate. If the analytical

20 model is formulated using a numerical method, a comparison of the

21 model to the data can be difficult due to dispersion properties

22 of the materials. These methods do not take into account large

23 compressive forces.

24   In the prior art, some efforts have been made to measure

25 material properties under large pressures. These methods consist

1 of placing materials in pressurized settings, insonifying them,

2 and then measuring their response. These methods are difficult

3 because they have to be conducted under great atmospheric

4 pressure that can adversely affect the instrumentation. Safety

5 issues can also arise in connection with laboratory testing at

6 extreme pressures. Finally, a mass loaded long thin rod has been

7 studied with respect to the bar wavespeed and corresponding

8 Young's modulus; however, this work does not investigate shear

9 motion.

10   Accordingly, there is a need for a method of measuring

11 mechanical properties of slab-shaped materials placed under

12 pressure.

13

14   SUMMARY OF THE INVENTION

15   Accordingly, in this invention, a method to measure the

16 complex frequency-dependent dilatational and shear wavenumbers of

17 a material under a static compressional force is provided. The

18 material is first vibrated in both vertical and horizontal

19 directions while obtaining transfer functions in each direction.

20 The two transfer functions are combined with a theoretical model

21 to estimate a dilatational wavenumber and a shear wavenumber.

22 The wavenumbers can be combined to give the complex dilatational

23 wavespeed, complex shear wavespeed, complex Lamé constants,

24 complex Young's modulus, complex shear modulus, and complex

25 Poisson's ratio.

BRIEF DESCRIPTION OF THE DRAWINGS

2   A more complete understanding of the invention and many of

3   the attendant advantages thereto will be readily appreciated as

4   the same becomes better understood by reference to the following

5   detailed description when considered in conjunction with the

6   accompanying drawings wherein:

7   FIG. 1 shows apparatus for measurement of transfer functions

8   in a vertical direction according to the current invention;

9   FIG. 2 shows apparatus for measurement of transfer functions

10   in a horizontal direction according to the current invention;

11   FIG. 3 is a diagram of the coordinate system of used with a

12   test specimen in the model;

13   FIG. 4A is a plot of the transfer function magnitude versus

14   input frequency for the vertical direction test;

15   FIG. 4B is a plot of the transfer function phase angle

16   versus input frequency for the vertical direction test;

17   FIG. 5A is a plot of the transfer function magnitude versus

18   input frequency for the horizontal direction test;

19   FIG. 5B is a plot of the transfer function phase angle

20   versus input frequency for the horizontal direction test;

21   FIG. 6 is a contour plot of the absolute value of the

22   dilatational wavenumber on an real-imaginary coordinate system of

23   the dilatational wavenumbers at 2000 Hz;

1   FIG. 7 is a contour plot of the absolute value of the

2   dilatational wavenumber on an real-imaginary coordinate system of

3   the dilatational wavenumbers at 5000 Hz;

4   FIG. 8A is a plot of the real dilatational wavenumber versus

5   frequency;

6   FIG. 8B is a plot of the imaginary dilatational wavenumber

7   versus frequency; and

8   FIG. 9A is a plot of the real shear wavenumber versus

9   frequency;

10   FIG. 9B is a plot of the imaginary shear wavenumber versus

11   frequency;

12   FIG. 10 is a plot of the real and imaginary Young's modulus

13   versus frequency;

14   FIG. 11 is a plot of the real and imaginary shear modulus

15   versus frequency; and

16   FIG. 12 is a plot of the Poisson's ratio versus frequency.

17

18   DESCRIPTION OF THE PREFERRED EMBODIMENT

19   The test procedure consists of vibrating a mass-loaded,

20   slab-shaped test specimen 10 with a shaker 12 in two different

21   directions, vertical 14A and horizontal 14B, as shown in FIGS. 1

22   and 2, respectively. It is noted that the load mass 16 attached

23   to the top of the test specimen 10 must be sufficiently stiffer

24   than the specimen 10 that it can be modeled as lumped parameter

25   expression rather than a continuous media system. A typical

1 example would be a steel load mass 16 attached above a rubber-

2 like material test specimen 10. This example results in a ratio

3 between the two stiffnesses of greater than 100. Lower ratios

4 result in less accurate estimations. Vibrating the combined

5 specimen 10 and load mass 16 causes different waveforms to

6 propagate in the specimen 10. The inverse method developed here

7 allows for the data from the experiments to be manipulated so

8 that the complex dilatational and shear wavenumbers can be

9 measured for the specimen 10. This test is usually done at

10 multiple frequencies (swept sine) so any frequency dependencies

11 can be identified and measured. Input vibration data is

12 collected from the shaker 12. A sensor 18 is mounted on load

13 mass 16 and another sensor 20 is mounted on shaker 12 for

14 collecting transfer function data. In FIG. 1, the test is set up

15 for monitoring the vertical transfer function. FIG. 2 shows the

16 test as set up for monitoring the horizontal transfer function.

17 Sensors 18 and 20 should be oriented properly to capture the

18 motion being measured. Other test configurations using

19 directions other than vertical and horizontal are possible;

20 however, the test setups shown are preferred for ease of set up

21 and calculation. These sensors 18 can be either accelerometers

22 that record accelerations, or laser velocimeters that record

23 velocities. In the swept sine mode, transfer functions of

24 acceleration divided by acceleration or velocity divided by

25 velocity are both equal to displacement divided by displacement.

1 The time domain data collected from the sensors 18 and 20 are

2 Fourier transformed into the frequency domain and then recorded

3 as complex transfer functions, typically using a spectrum

4 analyzer 22.

5   The motion of the test specimen shown in FIGS. 1 and 2 is

6 governed by the equation:

7

8      $$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla\nabla\bullet\mathbf{u} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2} \quad ,$$ (1)

9

10 where λ and μ are the complex Lamé constants (N/m^2);

11   ρ is the density (kg/m^3);

12   t is time (s);

13      $\bullet$ denotes a vector dot product; and

14      $\mathbf{u}$ is the Cartesian coordinate displacement vector of the

15 material.

16   The coordinate system of the test configuration is shown in

17 FIG. 3. Note that using this orientation results in $b = 0$ and a

18 having a value less than zero. The thickness of the specimen, h,

19 is a positive value. Equation (1) is manipulated by writing the

20 displacement vector $\mathbf{u}$ as

21

22      $$\mathbf{u} = \begin{Bmatrix} u_x(x,y,z,t) \\ u_y(x,y,z,t) \\ u_z(x,y,z,t) \end{Bmatrix} \quad ,$$ (2)

1 where x is the location along the plate (m), y is the location

2 into the plate (m), and z is the location normal to the plate

3 (m), as shown in FIG. 3. The symbol ∇ is the gradient vector

4 differential operator written in three-dimensional Cartesian

5 coordinates as

6

7
$$\nabla = \frac{\partial}{\partial x} i_x + \frac{\partial}{\partial y} i_y + \frac{\partial}{\partial z} i_z \quad , \tag{3}$$

8

9 with i_x denoting the unit vector in the x-direction, i_y denoting

10 the unit vector in the y-direction, and i_z denoting the unit

11 vector in the z-direction; ∇^2 is the three-dimensional Laplace

12 operator operating on vector $\mathbf{u}$ as

13

14
$$\nabla^2 \mathbf{u} = \nabla^2 u_x i_x + \nabla^2 u_y i_y + \nabla^2 u_z i_z \quad , \tag{4}$$

15

16 and operating on scalar u as

17

18
$$\nabla^2 u_{x,y,z} = \nabla \bullet \nabla u_{x,y,z} = \frac{\partial^2 u_{x,y,z}}{\partial x^2} + \frac{\partial^2 u_{x,y,z}}{\partial y^2} + \frac{\partial^2 u_{x,y,z}}{\partial z^2} \quad ; \tag{5}$$

19

20 and the term $\nabla \bullet \mathbf{u}$ is called the divergence and is equal to

21

22
$$\nabla \bullet \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \quad . \tag{6}$$

1

2 The displacement vector **u** is written as

3

4 $$\mathbf{u} = \nabla\phi + \nabla \times \bar{\psi} \quad , \tag{7}$$

5

6 where ϕ is a dilatational scalar potential, $\times$ denotes a vector

7 cross product, and $\bar{\psi}$ is an equivoluminal vector potential

8 expressed as

9

10 $$\bar{\psi} = \begin{Bmatrix} \psi_x(x,y,z,t) \\ \psi_y(x,y,z,t) \\ \psi_z(x,y,z,t) \end{Bmatrix} \quad . \tag{8}$$

11

12 The problem is formulated as a two-dimensional system, thus $y \equiv 0$,

13 $u_y(x,y,z,t) \equiv 0$, and $\partial(\cdot)/\partial y \equiv 0$. Expanding equation (7) and breaking

14 the displacement vector into its individual nonzero terms yields

15

16 $$u_x(x,z,t) = \frac{\partial\phi(x,z,t)}{\partial x} - \frac{\partial\psi_y(x,z,t)}{\partial z} \tag{9}$$

17

18 and

19

20 $$u_z(x,z,t) = \frac{\partial\phi(x,z,t)}{\partial z} + \frac{\partial\psi_y(x,z,t)}{\partial x} \quad . \tag{10}$$

1   Equations (9) and (10) are next inserted into equation (1),

2   which results in

3

4    $$c_d^2 \nabla^2 \phi(x,z,t) = \frac{\partial^2 \phi(x,z,t)}{\partial t^2}$$ (11)

5

6   and

7

8    $$c_s^2 \nabla^2 \psi_y(x,z,t) = \frac{\partial^2 \psi_y(x,z,t)}{\partial t^2}$$ (12)

9

10   where equation (11) corresponds to the dilatational component and

11   equation (12) corresponds to the shear component of the

12   displacement field.   Correspondingly, the constants c_d and c_s are

13   the complex dilatational and shear wave speeds, respectively, and

14   are determined by

15

16    $$c_d = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$ (13)

17

18   and

19

20    $$c_s = \sqrt{\frac{\mu}{\rho}} \quad .$$ (14)

1 The relationship of the Lamé constants to the Young's and shear

2 moduli is shown as

3

4
$$\lambda = \frac{E\upsilon}{(1+\upsilon)(1-2\upsilon)}$$
(15)

5

6 and

7

8
$$\mu = G = \frac{E}{2(1+\upsilon)} \quad,$$
(16)

9

10 where E is the complex Young's modulus (N/m^2), G is the complex

11 shear modulus (N/m^2), and υ is the Poisson's ratio of the

12 material (dimensionless).

13   The conditions of infinite length and steady-state response

14 are now imposed, allowing the scalar and vector potential to be

15 written as

16

17
$$\phi(x,z,t) = \Phi(z)\exp(ikx)\exp(i\omega t)$$
(17)

18

19 and

20

21
$$\psi_y(x,z,t) = \Psi(z)\exp(ikx)\exp(i\omega t)$$
(18)

1 where i is the square root of -1, ω is frequency (rad/s), and k

2 is wavenumber with respect to the x axis (rad/m). Inserting

3 equation (17) into equation (11) yields

4

5
$$\frac{d^2\Phi(z)}{dz^2} + \alpha^2\Phi(z) = 0 \quad , \tag{19}$$

6

7 where

8

9
$$\alpha = \sqrt{k_d^2 - k^2} \quad , \tag{20}$$

10

11 and

12

13
$$k_d = \frac{\omega}{c_d} \quad . \tag{21}$$

14

15 Inserting equation (18) into equation (12) yields

16

17
$$\frac{d^2\Psi(z)}{dz^2} + \beta^2\Psi(z) = 0 \quad , \tag{22}$$

18

19 where

20

21
$$\beta = \sqrt{k_s^2 - k^2} \quad , \tag{23}$$

1 and

2

3
$$k_s = \frac{\omega}{c_s} \quad .$$
(24)

4

5 The solution to equation (19) is

6

7
$$\Phi(z) = A(k,\omega)\exp(i\alpha z) + B(k,\omega)\exp(-i\alpha z) \quad ,$$
(25)

8

9 and the solution to equation (22) is

10

11
$$\Psi(z) = C(k,\omega)\exp(i\beta z) + D(k,\omega)\exp(-i\beta z) \quad ,$$
(26)

12

13 where $A(k,\omega)$, $B(k,\omega)$, $C(k,\omega)$, and $D(k,\omega)$ are wave response

14 coefficients that are determined below. The displacements can

15 now be written as functions of the unknown constants using the

16 expressions in equations (9) and (10). They are

17

$$u_z(x,z,t) = U_z(k,z,\omega)\exp(ikx)\exp(i\omega t)$$

18
$$= \{ \ i\alpha[A(k,\omega)\exp(i\alpha z) - B(k,\omega)\exp(-i\alpha z)] +$$
$$ik[C(k,\omega)\exp(i\beta z) + D(k,\omega)\exp(-i\beta z)] \ \} \ \exp(ikx) \ \exp(i\omega t) \ ,$$
(27)

19

20 and

$$u_x(x,z,t) = U_x(k,z,\omega)\exp(ikx)\exp(i\omega t)$$

1

$$= \{ \ ik[A(k,\omega)\exp(i\alpha z) + B(k,\omega)\exp(-i\alpha z)] - \tag{28}$$
$$i\beta[C(k,\omega)\exp(i\beta z) - D(k,\omega)\exp(-i\beta z)] \ \} \ \exp(ikx) \ \exp(i\omega t) \ .$$

2

3 Specific boundary conditions are now needed to individually solve

4 for the case of vertical and horizontal motion.   These are

5 formulated separately below.

6   For the case of vertical motion, the base at $z = a$ is

7 vibrated vertically using a shaker, as shown in FIG. 1.   Four

8 boundary conditions are necessary to formulate this problem.

9 Because the mass is attached to the material, the tangential

10 (horizontal) motion at the top of the plate ($z = b$) is zero and

11 this equation is written as

12

13      $$u_x(x,b,t) = 0 \ . \tag{29}$$

14

15 The normal stress at the top of the specimen is equal to and

16 opposite the load created by the mass in the z direction.   This

17 expression is

18

19 $$\tau_{zz}(x,b,t) = (\lambda + 2\mu)\frac{\partial u_z(x,b,t)}{\partial z} + \lambda\frac{\partial u_x(x,b,t)}{\partial x} = -M\frac{\partial^2 u_z(x,b,t)}{\partial t^2} \ , \tag{30}$$

1 where M is mass per unit area (kg/m^2) of the attached mass. The

2 tangential motion at the bottom of the plate ($z = a$) is zero and

3 this equation is written as

4

5 $$u_x(x,a,t) = 0 \ ,$$ (31)

6

7 and the normal motion at the bottom of the plate is prescribed as

8 a system input. This expression is

9

10 $$u_z(x,a,t) = U_0 \exp(i\omega t) \ .$$ (32)

11

12 Assembling equations (1) – (32) and letting $b = 0$ yields the

13 four-by-four system of linear equations that model the system.

14 They are

15

16 $$\mathbf{Ax = b} \ ,$$ (33)

17

18 where the entries of equation (33) are

19

20 $$A_{11} = ik \ ,$$ (34)

21

22 $$A_{12} = A_{11} \ ,$$ (35)

23

24 $$A_{13} = -i\beta \ ,$$ (36)

$$A_{14} = -A_{13} \quad , \tag{37}$$

$$A_{21} = -\alpha^2\lambda - 2\alpha^2\mu - \lambda k^2 - iM\omega^2\alpha \quad , \tag{38}$$

$$A_{22} = -\alpha^2\lambda - 2\alpha^2\mu - \lambda k^2 + iM\omega^2\alpha \quad , \tag{39}$$

$$A_{23} = -2k\beta\mu - iM\omega^2 k \quad , \tag{40}$$

$$A_{24} = 2k\beta\mu - iM\omega^2 k \quad , \tag{41}$$

$$A_{31} = A_{11}\exp(i\alpha a) \quad , \tag{42}$$

$$A_{32} = A_{11}\exp(-i\alpha a) \quad , \tag{43}$$

$$A_{33} = A_{13}\exp(i\beta a) \quad , \tag{44}$$

$$A_{34} = -A_{13}\exp(-i\beta a) \quad , \tag{45}$$

$$A_{41} = i\alpha\exp(i\alpha a) \quad , \tag{46}$$

$$A_{42} = -i\alpha\exp(-i\alpha a) \quad , \tag{47}$$

$$A_{43} = ik\exp(i\beta a) \quad , \tag{48}$$

16

$$A_{44} = ik \exp(-i\beta a) \quad , \tag{49}$$

$$x_{11} = A(k,\omega) \quad , \tag{50}$$

$$x_{21} = B(k,\omega) \quad , \tag{51}$$

$$x_{31} = C(k,\omega) \quad , \tag{52}$$

$$x_{41} = D(k,\omega) \quad , \tag{53}$$

$$b_{11} = 0 \quad , \tag{54}$$

$$b_{21} = 0 \quad , \tag{55}$$

$$b_{31} = 0 \quad , \tag{56}$$

and

$$b_{41} = U_0 \quad . \tag{57}$$

Using equations (34) - (57) the solution to the constants $A(k,\omega)$, $B(k,\omega)$, $C(k,\omega)$, and $D(k,\omega)$ can be calculated at each specific wavenumber and frequency using

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \quad . \tag{58}$$

1 Noting that for vertical motion, $k = 0$, and using the

2 coefficients from equation (58), the transfer function between

3 the vertical base displacement and the vertical mass displacement

4 can be written as

5

6
$$T_1(\omega) = \frac{1}{R_1(\omega)} = \frac{U_z(0,b,\omega)}{U_0} = \frac{1}{\cos(k_d h) - \left(\frac{M}{\rho}\right) k_d \sin(k_d h)} \quad , \tag{59}$$

7

8 where $T_1(\omega)$ or $R_1(\omega)$ correspond to the data from the vertical

9 motion experiment.

10   The next step is to solve the inverse problem for vertical

11 motion. This involves using the experimental data and equation

12 (59) to estimate the dilatational wavenumber. Equation (59) can

13 be rewritten as

14

15
$$f(k_d) = 0 = \cos(k_d h) - \left(\frac{M}{\rho}\right) k_d \sin(k_d h) - R_1 \tag{60}$$

16

17 where the problem now becomes finding the zeros of the right-hand

18 side of equation (60), or, in the presence of actual data that

19 contains noise, finding the relative minima of the right-hand

20 side of equation (60) and determining which relative minimum

21 corresponds to dilatational wave propagation and which relative

22 minima are extraneous. Because equation (60) has a number of

23 relative minima, zero finding algorithms are not applied to this

18

1   function, as they typically do not find all of the minima
2   locations and are highly dependent on initial starting locations.
3   The best method to find all of the minima locations is by
4   plotting the absolute value of the right-hand side of equation
5   (60) as a surface with the real part of dilatational wavenumber
6   k_d on one axis and the imaginary part of k_d on the other axis.
7   In order to do this, the user should start at a low frequency
8   where the aliasing minimum has not yet appeared. In the specific
9   example shown herein, this is below 3850 Hz for the dilatational
10 wave and below 1550 Hz for the shear wave. At these lower
11 frequencies, the minimum furthest to the left will correspond to
12 dilatational wave propagation. As the frequency increases,
13 extraneous minima will appear to the left of the minimum that
14 corresponds to dilatational wave propagation, however, the wave
15 propagation minimum will always be close to the previous test
16 frequency wave propagation minimum provided that the frequency
17 increments are relatively small. At a resolution of 0.5 rad/m
18 for the materials in the example herein, this requires a
19 frequency increment of 37.3 Hz for the dilatational measurement
20 and 14.4 Hz for the shear measurement. Different test specimens
21 and top masses require different increments. Additionally, the
22 real part of the wavenumber is a monotonically increasing
23 function with respect to frequency, so at each increase in
24 frequency, the new wavenumber to be estimated has to be greater
25 than the old wavenumber that was previously estimated. This

19

1   process is further illustrated as related to the discussion

2   concerning FIG. 6 and FIG. 7 below.

3   For the case of horizontal motion, the base at $z = a$ is

4   vibrated horizontally using a shaker, as shown in FIG. 2. Four

5   boundary conditions are necessary to formulate this problem.

6   Because the mass is attached to the material, the shear

7   (tangential) stress at the top of the plate is equal to opposite

8   the load created by the mass in the x direction. This expression

9   is

10

11   $$\tau_{zx}(x,b,t) = \mu \left[\frac{\partial u_x(x,b,t)}{\partial z} + \frac{\partial u_z(x,b,t)}{\partial x} \right] = -M \frac{\partial^2 u_x(x,b,t)}{\partial t^2} \ , \qquad (61)$$

12

13   where M is mass per unit area (kg/m^2) of the attached mass. The

14   normal motion at the top of the plate ($z = b$) is zero and this

15   equation is written as

16

17   $$u_z(x,b,t) = 0 \ . \qquad\qquad (62)$$

18

19   The tangential motion at the bottom of the plate ($z = a$) is

20   prescribed as a system input and this equation is written as

21

22   $$u_x(x,a,t) = V_0 \exp(i\omega t) \ , \qquad\qquad (63)$$

1 and the normal motion at the bottom of the plate is zero. This

2 expression is

3

4    $$u_z(x,a,t)=0 \quad .$$ (64)

5

6   Assembling equations (1) − (28) and (62) − (64) and letting

7 $b = 0$ yields the four-by-four system of linear equations that

8 model the system. They are

9

10    $$\mathbf{Ax}=\mathbf{b} \quad ,$$ (65)

11

12 where the entries of equation (61) are

13

14    $$A_{11} = -2\mu k\alpha - i\omega^2 Mk \quad ,$$ (66)

15

16    $$A_{12} = 2\mu k\alpha - i\omega^2 Mk \quad ,$$ (67)

17

18    $$A_{13} = \mu\beta^2 - \mu k^2 + i\omega^2 M\beta \quad ,$$ (68)

19

20    $$A_{14} = \mu\beta^2 - \mu k^2 - i\omega^2 M\beta \quad ,$$ (69)

21

22    $$A_{21} = i\alpha \quad ,$$ (70)

23

24    $$A_{22} = -A_{21} \quad ,$$ (71)

21

1

$$A_{23} = ik \quad , \tag{72}$$

3

$$A_{24} = A_{23} \quad , \tag{73}$$

5

$$A_{31} = A_{23} \exp(i\alpha a) \quad , \tag{74}$$

7

$$A_{32} = A_{23} \exp(-i\alpha a) \quad , \tag{75}$$

9

$$A_{33} = -i\beta \exp(i\beta a) \quad , \tag{76}$$

11

$$A_{34} = i\beta \exp(-i\beta a) \quad , \tag{77}$$

13

$$A_{41} = A_{21} \exp(i\alpha a) \quad , \tag{78}$$

15

$$A_{42} = -A_{21} \exp(-i\alpha a) \quad , \tag{79}$$

17

$$A_{43} = A_{23} \exp(i\beta a) \quad , \tag{80}$$

19

$$A_{44} = A_{23} \exp(-i\beta a) \quad , \tag{81}$$

21

22

$$x_{11} = A(k,\omega) \quad , \tag{82}$$

24

$$x_{21} = B(k,\omega) \quad , \tag{83}$$

1

2 $$x_{31} = C(k,\omega) \quad , \tag{84}$$

3

4 $$x_{41} = D(k,\omega) \quad , \tag{85}$$

5

6

7 $$b_{11} = 0 \quad , \tag{86}$$

8

9 $$b_{21} = 0 \quad , \tag{87}$$

10

11 $$b_{31} = V_0 \quad , \tag{88}$$

12 and

13

14 $$b_{41} = 0 \quad . \tag{89}$$

15

16 Using equations (67) - (89) the solution to the constants $A(k,\omega)$,

17 $B(k,\omega)$, $C(k,\omega)$, and $D(k,\omega)$ can be calculated at each specific

18 wavenumber and frequency using

19

20 $$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \quad . \tag{90}$$

21

22 Noting that for horizontal motion, $k = 0$, and using the

23 coefficients from equation (90), the transfer function between

24 the horizontal base displacement and the horizontal mass

25 displacement can be written as

$$T_2(\omega) = \frac{1}{R_2(\omega)} = \frac{U_x(0,b,\omega)}{V_0} = \frac{1}{\cos(k_s h) - \left(\dfrac{M}{\rho}\right) k_s \sin(k_s h)} \;. \tag{91}$$

where $T_2(\omega)$ or $R_2(\omega)$ correspond to the data from the horizontal motion experiment.

The next step is to solve the inverse problem for horizontal motion. This involves using the data and equation (91) to estimate the shear wavenumber. Equation (91) can be rewritten as

$$f(k_s) = 0 = \cos(k_s h) - \left(\frac{M}{\rho}\right) k_s \sin(k_s h) - R_2 \;. \tag{92}$$

It is noted that this equation is identical, except for the subscripts, to equation (60). The shear wavenumber is estimated using the same procedure that was used to estimate the dilatational wavenumber above.

The material properties can be determined from the wavenumbers. First, the dilatational and shear wavespeeds are determined using

$$c_d = \frac{\omega}{k_d} \tag{93}$$

and

24

1 $$c_s = \frac{\omega}{k_s} \ ,$$ (94)

2

3 respectively. The Lamé constants are calculated using equations

4 (13) and (14) written as

5

6 $$\mu = \rho c_s^2$$ (95)

7

8 and

9

10 $$\lambda = \rho c_d^2 - 2\rho c_s^2 \ .$$ (96)

11 Poisson's ratio is then calculated using

12

13 $$\upsilon = \frac{\lambda}{2(\mu + \lambda)} \ .$$ (97)

14

15 Young's modulus can be calculated with

16

17 $$E = \frac{2\mu(2\mu + 3\lambda)}{2(\mu + \lambda)}$$ (98)

18

19 and the shear modulus can be determined using

20

21 $$G \equiv \mu \ .$$ (99)

22   The above measurement method can be simulated by means of a

23 numerical example. Soft rubber-like material properties of the

1 test specimen are used in this simulation. The material has a

2 Young's modulus E of $[(1e8-i2e7)+(5e3f-i3e2f)]$ N/m^2 where f is

3 frequency in Hz, Poisson's ratio υ is equal to 0.40

4 (dimensionless), density ρ is equal to 1200 kg/m^3, and a

5 thickness h of 0.1 m. The top mass is a 0.0254 m (1 inch) steel

6 plate that has a mass per unit area value M of 199 kg/m^2. FIG. 4

7 is a plot of the transfer function of the system for vertical

8 motion and corresponds to equation (59). FIG. 5 is a plot of the

9 transfer function of the system for horizontal motion and

10 corresponds to equation (91). In FIGS. 4 and 5, the top plot is

11 the magnitude in decibels and the bottom plot is the phase angle

12 in degrees.

13   FIG. 6 is a contour plot of the absolute value of equation

14 (60) expressed in decibels versus real dilatational wavenumber on

15 the x axis and imaginary dilatational wavenumber on the y axis at

16 2000 Hz. The estimated dilatational wavenumber, read directly

17 from the plot at the location the minimum value appears and

18 marked with an arrow, is 27.89 + 2.61i rad/m. The actual value

19 of the dilatational wavenumber is 27.99 + 2.60i rad/m, which is

20 slightly different from the estimated value due to the surface

21 discretization of equation (60). FIG. 7 is a contour plot of

22 equation (60) at 5000 Hz. At this frequency, an extraneous

23 minimum appears on the left-hand side of the plot. However,

24 because the real part of the wavenumber must be increasing with

1   increasing frequency, the minimum corresponding to dilatational

2   wave propagation is located at the arrow marked spot and is equal

3   to 65.86 + 5.60i rad/m, as compared to an actual value of 65.77 +

4   5.62i rad/m. Again, the difference between the two values can be

5   attributed to the discretization of the surface.

6   FIG. 8 is plot of actual (solid line) and estimated (x

7   symbols) dilatational wavenumber versus frequency. FIG. 9 is

8   plot of actual (solid line) and estimated (+ symbols) shear

9   wavenumber versus frequency. In FIGS. 8 and 9, the top plot is

10   the real part of the wavenumber and the bottom part is the

11   imaginary part of the wavenumber. FIG. 10 is a plot of actual

12   (solid line) and estimated (real part - x symbols, imaginary part

13   - o symbols) Young's modulus versus frequency. FIG. 11 is a plot

14   of actual (solid line) and estimated (real part - x symbols,

15   imaginary part - o symbols) shear modulus versus frequency. In

16   FIGS. 10 and 11, the imaginary part of the modulus all have a

17   negative sign but are depicted with positive signs for plotting

18   purposes. FIG. 12 is a plot of actual (solid line) and estimated

19   (square symbols) of the real part of Poisson's ratio versus

20   frequency. Because the numerical example is formulated using a

21   Poisson's ratio that is strictly real, no imaginary component is

22   shown in this plot. Imaginary values of Poisson's ratio are

23   possible and have been shown to theoretically exist.

24   This invention gives the ability to estimate complex

25   dilatational and shear wavespeeds of a material that is slab-

27

1   shaped and subjected to compressive forces.   It also allows

2   estimation of complex Lamé constants of a material that is slab-

3   shaped and subjected to compressive forces.   Complex Young's and

4   shear moduli of a material that is slab-shaped and subjected to

5   compressive forces can be estimated using this invention.   The

6   invention also allows estimation of the complex Poisson's ratio

7   of a material that is slab-shaped and subjected to compressive

8   forces.   The advantage of this patent is that it does not require

9   a testing configuration that has to be pressurized.

10   Obviously many modifications and variations of the present

11   invention may become apparent in light of the above teachings.

12   In light of the above, it is therefore understood that within the

13   scope of the appended claims, the invention may be practiced

14   otherwise than as specifically described.